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Hidden Measurement Model for Pure and Mixed States of Quantum Physics in Euclidean Space

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We propose a representation of quantum mechanics where all pure and mixed states of a n-dimensional quantum entity are represented as points of a subset of a n^2-dimensional real space. We introduce the general measurements of quantum mechanics on this entity, determined by sets of mutual orthogonal points of the representation space. Within this framework we construct a hidden measurement model for an arbitrary finite dimensional quantum entity.

1. INTRODUCTION

In Aerts (1986), it is shown that it is possible to find the quantum structure originating in the presence of a lack of knowledge about the interaction between the measuring apparatus and the physical entity under study. Aerts has expressed this idea in the following way:

1. To each real measurement e there corresponds a collection of deterministic measurements e_λ, $\lambda \in \Lambda$, and these deterministic measurements are called "hidden measurements" in analogy with the "hidden variables."

2. When a measurement e is performed on an entity S in a pure state p, then one of the hidden measurements e_λ takes place. The probability finds its origin in the lack of knowledge about which one of the hidden measurements effectively takes place.

This approach is not in contradiction with the no-go theorems about hidden variables (all of them inspired by the von Neumann proof), since the hidden variables in the hidden measurement approach are contextual by definition. It is important to notice that the state p is not dependent on the parameter λ. Analogously, the selection of one lambda is independent of the

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state of the system. It is these two restrictions which distinguish between a general hidden variable model and a hidden measurement model.

In the same paper, Aerts introduces a concrete model, now referred to as the "elastic-model" (see Aerts, 1994), which entails a representation of a hidden measurement model for the spin of a spin-1/2 quantum entity in a three-dimensional Euclidean space. It was an open question whether such an explicit real-space hidden-measurement representation could be constructed for a quantum entity described in an arbitrary n-dimensional complex Hilbert space.

We show in this paper that the answer to this question is affirmative. It also poses no problem to consider the set of mixed states of this entity as state space. To do this we use a Euclidean representation of quantum mechanics as introduced in Coecke (1994). In Sections 2 and 3 of this paper we summarize some aspects of this representation. In Section 4 we describe the generalized elastic model.

2. PURE STATES, MIXED STATES, AND PURE MEASUREMENTS

We denote by Σ^{pure} the set of pure states of a quantum entity S. This means that every state $p \in \Sigma^{\text{pure}}$ can be represented as a ray $\hat{\psi}_p$ in a Hilbert space $\mathcal{H}$. If n is the dimension of the Hilbert space, we will state this by writing Σ_n^{pure}. The n-dimensional Hilbert space itself will be denoted by $\mathcal{H}_n$. We will always consider $j > i$ in expressions containing $r_i r_j$. We also consider $r_i \in [0, \infty[$ and $\theta_i \in [0, 2\pi]$. With $\theta_{ij} = \Sigma_{i<k\leq j}\, \theta_k$ we introduce Ξ_n as the following set of points of the n^2-dimensional real space:

$$\begin{aligned}\Xi_n = \{(&\sqrt{2}\, r_1 r_2 \cos \theta_2, \sqrt{2}\, r_1 r_2 \sin \theta_2, \dots, \\ &\sqrt{2}\, r_i r_j \cos \theta_{ij}, \sqrt{2}\, r_i r_j \sin \theta_{ij}, \dots, \\ &\sqrt{2}\, r_{n-1} r_n \cos \theta_n, \sqrt{2}\, r_{n-1} r_n \sin \theta_n, \\ &r_1^2, \dots, r_i^2, \dots, r_n^2) \mid \sum_{1\leq i\leq n} r_i^2 = 1\}\end{aligned} \tag{1}$$

Define $\mu_n: \mathcal{H}_n \to \Xi_n$ by

$$\begin{aligned}\mu_n(\psi) = \frac{1}{|\psi|^2}(&\sqrt{2}\,\mathrm{Re}(\psi_1\overline{\psi}_2), \sqrt{2}\,\mathrm{Im}(\psi_1\overline{\psi}_2), \dots, \\ &\sqrt{2}\,\mathrm{Re}(\psi_i\overline{\psi}_j), \sqrt{2}\,\mathrm{Im}(\psi_i\overline{\psi}_j), \dots, \\ &\sqrt{2}\,\mathrm{Re}(\psi_{n-1}\overline{\psi}_n), \sqrt{2}\,\mathrm{Im}(\psi_{n-1}\overline{\psi}_n), \psi_1\overline{\psi}_1, \dots, \psi_i\overline{\psi}_i, \dots, \psi_n\overline{\psi}_n)\end{aligned} \tag{2}$$

If we introduce $\mathcal{S}_n^{\text{pure}}$: $\Sigma_n^{\text{pure}} \to \mathcal{H}_n$ which maps a quantum state p which corresponds with the ray $\hat{\psi}_p$ onto a representative vector ψ_p, we can define $\mathcal{R}_n^{\text{pure}}$: $\Sigma_n^{\text{pure}} \to \Xi_n$ as $\mu_n \circ \mathcal{S}_n^{\text{pure}}$. In Coecke (1994) we proved that this map is bijective.

Proposition 1. The set of points Ξ_2 are the points of S^2, the 2-dimensional sphere.

Proof. Let us introduce a translation T on Ξ_2 with the vector $(0, 0, -1/2, -1/2)$, a scaling K with a factor $\sqrt{2}$, and a rotation O of the last two components over an angle of $\pi/4$. We obtain

$$\begin{aligned}&[O \circ K \circ T](\Xi_2)\\ &\quad = \{(2r_1 r_2 \cos\theta_2, 2r_1 r_2 \sin\theta_2, r_1^2 - r_2^2, 0) \mid r_1^2 + r_2^2 = 1\}\end{aligned}$$

For $x = (x_1, x_2, x_3, 0) \in [O \circ K \circ T](\Xi_2)$ we have

$$\begin{aligned}|x|^2 &= x_1^2 + x_2^2 + x_3^2\\ &= 4r_1^2 r_2^2 \cos^2\theta_2 + 4r_1^2 r_2^2 \sin^2\theta_2 + r_1^4 + r_2^4 - 2r_1^2 r_2^2\\ &= 2r_1^2 r_2^2 + r_1^4 + r_2^4\\ &= (r_1^2 + r_2^2)^2 = 1 \quad \blacksquare\end{aligned}$$

In quantum mechanics a measurement e performed on an entity is represented by a self-adjoint operator H_e on an Hilbert space $\mathcal{H}$. We know that every self-adjoint operator H_e is completely determined by its spectral measure E: $\mathcal{B}(\sigma_e) \to \mathcal{E}_{\mathcal{H}}$, where to every A in $\mathcal{B}(\sigma_e)$, the collection of Borel sets of the spectrum σ_e of the operator H_e, there corresponds an orthogonal projection E_A in $\mathcal{E}_{\mathcal{H}}$, the set of orthogonal projections. For a finite-dimensional Hilbert space $\mathcal{H}_n$ we can write a self-adjoint operator H_e with $\{\hat{\psi}_1, \ldots, \hat{\psi}_n\}$ as a set of mutual orthogonal eigenrays and $\{e_1, \ldots, e_n\}$ (some of them may be equal) as corresponding eigenvalues as follows

$$H_e = \sum_i e_i E_{\hat{\psi}_i} \tag{3}$$

where $E_{\hat{\psi}_i}$ is the projector on the ray $\hat{\psi}_i$. Following equation (3), the projectors on one-dimensional subspaces of the Hilbert space are the building blocks of a general measurement.

Definition 1. If a measurement e can be represented by $E_{\hat{\psi}_q}$, the projector on a ray of the Hilbert space $\mathcal{H}$, we call it a pure measurement. Such a pure measurement will be denoted by e_q.

Definition 2. If the entity S is in a state $p \in \Sigma$, and the measurement e is performed, the state of the entity changes. The probability that the state p

is changed by this measurement into the state q shall be denoted by $P_e(q \mid p)$ and we call $P_e: \Sigma \times \Sigma \rightarrow [0, 1]$ the transition probability by e in Σ.

Lemma 1. Let $\psi, \phi \in \mathcal{H}_n$ such that $|\psi| = |\phi| = 1$. Then $|\langle\psi|\phi\rangle|^2 = \mu_n(\psi) \cdot \mu_n(\phi)$.

Proof. Write $\psi = (\psi_1, \ldots, \psi_n)$ and $\phi = (\phi_1, \ldots, \phi_n)$. Then

$$
\begin{aligned}
&|\langle\psi|\phi\rangle|^2 \\
&\quad = |\psi_1\overline{\phi}_1 + \cdots + \psi_n\overline{\phi}_n|^2 \\
&\quad = \sum_{i,j} \psi_i\overline{\psi}_j\phi_j\overline{\phi}_i \\
&\quad = \sum_{i} \psi_i\overline{\psi}_i\phi_i\overline{\phi}_i + \sum_{i<j} \psi_i\overline{\psi}_j\phi_j\overline{\phi}_i + \sum_{i<j} \overline{\psi_i\overline{\psi}_j\phi_j\overline{\phi}_i} \\
&\quad = \sum_{i} \psi_i\overline{\psi}_i\phi_i\overline{\phi}_i + 2\sum_{i<j} [\mathrm{Re}(\psi_i\overline{\psi}_j)\,\mathrm{Re}(\phi_i\overline{\phi}_j) + \mathrm{Im}(\psi_i\overline{\psi}_j)\,\mathrm{Im}(\phi_i\overline{\phi}_j)] \\
&\quad = \sum_{i} \psi_i\overline{\psi}_i\phi_i\overline{\phi}_i + \sum_{i<j} \sqrt{2}\,\mathrm{Re}(\psi_i\overline{\psi}_j)\sqrt{2}\,\mathrm{Re}(\phi_i\overline{\phi}_j) \\
&\qquad + \sum_{i<j} \sqrt{2}\,\mathrm{Im}(\psi_i\overline{\psi}_j)\sqrt{2}\,\mathrm{Im}(\phi_i\overline{\phi}_j) \\
&\quad = \mu_n(\psi)\cdot\mu_n(\phi) \quad \blacksquare
\end{aligned}
$$

Theorem 1. If e_q is a pure measurement on an entity S in state $p \in \Sigma_n^{\mathrm{pure}}$, we have for $x_q = \mathcal{R}_n^{\mathrm{pure}}(q)$ and $x_p = \mathcal{R}_n^{\mathrm{pure}}(p)$: $P_{e_q}(q \mid p) = x_q \cdot x_p$.

Definition 3. Let $\{p_i \mid i \in I\} \subset \Sigma^{\mathrm{pure}}$, where I is a countable set of indices. Then $\rho: \{p_i \mid i \in I\} \rightarrow [0, 1]$ with $\Sigma_i\, \rho(p_i) = 1$ defines a probability measure on $\{p_i \mid i \in I\}$. We define a mixed state p_ρ by stating that the state of S is p_i with probability $\rho(p_i)$. The set of all mixed states of the entity S shall be denoted by Σ^{mix}.

Clearly $\Sigma^{\mathrm{pure}} \subset \Sigma^{\mathrm{mix}}$. A "mixed" state describes a situation with a lack of knowledge concerning the pure state of the system. As a direct consequence of Definition 3 we also have the following relation concerning the transition probabilities: $P_{e_q}(q \mid p_\rho) = \Sigma_i\, \rho(p_i) P_{e_q}(q \mid p_i)$.

Let $\overline{\Xi}_n$ be the convex closure of Ξ_n. We can now formulate the main theorem concerning the representation of the states of a quantum entity in Euclidean space. The proof of this theorem can be found in Coecke (1994).

Theorem 2. We can define a one-to-one map $\mathcal{R}_n^{\mathrm{mixed}}: \Sigma_n^{\mathrm{mixed}} \rightarrow \overline{\Xi}_n$ which maps every state $p_\rho \in \Sigma_n^{\mathrm{mixed}}$ onto a point x_ρ, the geometrical mean of the composing states in the mixed state $\Sigma_i\, \rho(p_i) x_{p_i}$, and which is such that we have following relation for the probabilities: $P_{e_q}(q \mid p_\rho) = x_q \cdot x_\rho$, and such that when we restrict the domain to Ξ_n we have $\mathcal{R}_n^{\text{mixed}}|_{\Xi_n} = \mathcal{R}_n^{\text{pure}}$.

3. GENERAL MEASUREMENTS

As expressed in equation (3), we can construct the measurements with n outcomes on the states represented in Ξ_n, with projectors $E_{\hat{\psi}}$, representing pure measurements. By assigning the same eigenvalue to different eigenstates, we can also construct the measurements with less than n outcomes. The probability for obtaining an outcome e_i will be denoted by $P_{e_i}(p)$.

Theorem 3. Let S be a quantum entity represented in $\mathcal{H}_n$. Let e be a measurement represented by a self-adjoint operator H_e with n different eigenvalues $\{e_1, \ldots, e_n\}$. Then there exists one set $\mathcal{X}_e = \{x_{e_1}, \ldots, x_{e_n}\}$ of mutual orthogonal points of Ξ_n such that for every state p_ρ of S, we have following relation for the probability for obtaining an outcome e_i: $P_{e_i}(p_\rho) = x_{e_i} \cdot x_\rho$.

Proof. Since all eigenvalues are different, there exists one set $\{\hat{\psi}_{e_1}, \ldots, \hat{\psi}_{e_n}\}$ of mutual orthogonal eigenrays corresponding to these eigenvalues, and thus a set $\{p_1, \ldots, p_n\}$ of corresponding eigenstates. Let x_{e_i} be $\mathcal{R}_n^{\text{pure}}(p_i) \in \Xi_n$. For $i \neq j$ we have $x_{e_i} \cdot x_{e_j} = |\langle \psi_{e_i} | \psi_{e_j} \rangle|^2 = 0$. For the probability we find

$$P_{e_i}(p_\rho) = P_e(p_i | p_\rho) = |\langle \psi_{e_i} | \psi_\rho \rangle|^2 = P_{e_{p_i}}(p_i | p_\rho) = x_{e_i} \cdot x_\rho$$

Unicity follows from, on the one hand, the fact that $\mathcal{R}_n^{\text{pure}}: \Sigma_n^{\text{pure}} \to \Xi_n$ is one to one such that $H \neq H' \Rightarrow e_H \neq e_{H'}$ and, on the other hand, the unicity of a state p_ρ that fulfills $P_e(p_i | p_\rho) = 1$, namely the state p_i itself. ■

Theorem 4. Let $\mathcal{X} = \{x_1, \ldots, x_k\}$ be a set of k mutual orthogonal points of Ξ_n, and thus representing $A_{\mathcal{X}}$, a subspace of $\mathcal{H}_n$ with $[\mu_n]^{-1}(\mathcal{X})$ as a base. If we define $x_{\mathcal{X}}$ by $x_{\mathcal{X}} = \Sigma_{i=1}^{i=k} x_i$, we have for the points in the pointwise representation of the subspace $x_{\mathcal{X}} \cdot x = 1$. Thus we have for $\widehat{xx_{\mathcal{X}}}$, the angle between x and $x_{\mathcal{X}}$, $\widehat{xx_{\mathcal{X}}} = \text{Arccos}(1/\sqrt{k})$.

Proof. We have

$$x \cdot x_{\mathcal{X}} = x \cdot \sum_{i=1}^{i=k} x_i = \sum_{i=1}^{i=k} (x \cdot x_i) = \sum_{i=1}^{i=k} |\langle \psi_x | \psi_{x_i} \rangle|^2 = 1 \quad (\psi_x \in A_{\mathcal{X}})$$

Since all the x_i are mutual orthogonal, we have that $|x_{\mathcal{X}}| = |\Sigma_{i=1}^{i=k} x_i| = \sqrt{k}$. ■

Theorem 5. Let A be a k-dimensional subspace of $\mathcal{H}_n$ and let us denote by $|\langle A | \psi \rangle|$ the modulus of the orthogonal projection of the vector ψ on the subspace A. Then there exists a unique point x_A such that for every possible orthonormal base $\{\psi_1, \ldots, \psi_k\}$ we have $x_A = \Sigma_{i=1}^{i=k} \mu_n(\psi_i)$ $(= \Sigma_{i=1}^{i=k} x_i)$. We also have $|\langle A|\psi\rangle|^2 = x_A \cdot x_\psi$ for every $\psi \in \mathcal{H}_n$ and thus x_A is representative for A in the way that a point x_ψ is representative for the ray determined by ψ.

Proof. We have to prove that for every other set $\mathcal{X}' = \{x'_1, \ldots, x'_k\}$ of mutual orthogonal points in $\mu_n(A)$ we still have that $\Sigma_{i=1}^{i=k} x'_i = x_{\mathcal{X}'} = x_{\mathcal{X}}$. Let k be the dimension of A. Because of Theorem 2 we know that there exist k^2 linear independent vectors in A, and hence a base of the k^2-dimensional real vector space (Theorem 4). Since both points $x_{\mathcal{X}'}$ and $x_{\mathcal{X}}$ have the same coordinates in this base (Theorem 4), namely all 1, we can only have a unique point x_A. Concerning the probabilities we have

$$|\langle A|\psi\rangle|^2 = \sum_{i=1}^{i=k} |\langle\psi_i|\psi\rangle|^2 = \sum_{i=1}^{i=k} (x_{\psi_i} \cdot x_\psi) = x_\psi \cdot \sum_{i=1}^{i=k} x_{\psi_i} = x_\psi \cdot x_A \quad \blacksquare$$

Definition 4. Let $\mathcal{A}_n$ be the set of all subspaces of the Hilbert space $\mathcal{H}_n$. We also introduce following set:

$$\Xi_{\mathcal{A}_n} = \{ \sum_{\psi_i \in \mathcal{X}_A} \mu_n(\psi_i) | A \in \mathcal{A}_n, \mathcal{X}_A \text{ base of } A \}$$

Then define $\mu_{\mathcal{A}_n}: \mathcal{A}_n \to \Xi_{\mathcal{A}_n}$ as the map which maps every subspace $A \in \mathcal{A}_n$ onto its representative point x_A.

The results of Theorems 3–5 lead us to the main theorem of this section.

Theorem 6. Let S be a quantum entity represented in $\mathcal{H}_n$. Let e be a measurement represented by a self-adjoint operator H_e with eigenvalues $e_1, \ldots, e_l$ and corresponding eigenspaces $A_1, \ldots, A_l$. Then there exists one set $\mathcal{X}_e = \{x_{e_1}, \ldots, x_{e_l}\}$ of mutual orthogonal points of $\mu_{\mathcal{A}_n}(\mathcal{A}_n)$ such that for every state p_ρ of S, we have $P_{e_i}(p_\rho) = x_{e_i} \cdot x_\rho$ as probability for obtaining an outcome e_i. For the points in $\mathcal{X}_e$ we have $\Sigma_{i=1}^{i=l} x_{e_i} = x_{\mathcal{H}_n}$.

4. A GENERAL HIDDEN MEASUREMENT MODEL FOR QM IN A FINITE-DIMENSIONAL HILBERT SPACE

Let $\Lambda = \{x | \Sigma_{1 \leq i \leq n} x_i = 1 \text{ and } 0 \leq x_i \leq 1\}$ be the set of points of the Euclidean simplex spanned by $\{h_1, \ldots, h_n\}$, a set of canonical base vectors. Let x be one point in Λ. A "selection" s for a given x consists in executing a deterministic process s_λ where $\lambda \in \Lambda$ is a uniformly distributed stochastic variable. We define s_λ as follows:

- Let Λ_i be the convex closure of $\{h_1, \ldots, h_{i-1}, x, h_{i+1}, \ldots, h_n\}$ and thus $\cup_i \Lambda_i = \Lambda$.

- If $\lambda \in \Lambda_i$, then x changes to h_i.
- The probability of λ being on the border of two regions Λ_i and Λ_j is zero. These situations will not contribute to global probabilistic results and as a consequence it will not be necessary to make any conventions for this case.

This concludes the description of s_λ and thus s itself.

Proposition 2. Consider a point $x = (x_1, \ldots, x_n) \in \Lambda$ with Λ defined as above. Let s be a selection mechanism. The probability for obtaining an outcome h_i is x_i.

The proof of this proposition can be found in Aerts (1986).

We will now describe the hidden measurement model for a measurement e with n different possible outcomes on the entity S described in $\mathcal{H}_n$. Let $x_\rho \in \Xi_n$ be representative for the state p_ρ of the entity. A measurement e will be defined by the following steps:

- Let $\{x_{e_1}, \ldots, x_{e_n}\}$ be the set of eigenstates of e. These n-points define an $(n - 1)$-dimensional Euclidean simplex.
- One can project x_ρ orthogonally on the n-dimensional subspace spanned by $\{x_{e_1}, \ldots, x_{e_n}\}$. This gives a point x'_ρ (this is in fact a first transition of the state in the space Ξ_n: $x_\rho \to x'_\rho$).
- We now perform a hidden measurement e_λ through the selection s_λ with x'_ρ as the initial state and $\{x_{e_1}, \ldots, x_{e_n}\}$ as the canonical base $\{h_1, \ldots, h_n\}$. Thus we find one unique final state and thus one unique outcome for this measurement (this is a second transition of the state in Ξ_n: $x'_\rho \to x_{\text{outcome}}$).

Theorem 7. The probability for obtaining a final state p_i represented by x_{e_i} in a measurement $e \in \mathcal{E}_n$ on an entity in a state p_ρ described by Ξ_n is the scalar product $x_\rho \cdot x_{e_i}$: $P_e(p_i|p_\rho) = x_\rho \cdot x_{e_i}$.

Proof. x'_ρ is the restriction of x_ρ to the linear subspace spanned by $\{x_{e_1}, \ldots, x_{e_n}\}$. In this subspace of outcome states we can express x'_ρ in the base $\{x_{e_1}, \ldots, x_{e_n}\}$, which gives us $(x'_\rho \cdot x_{e_1}, \ldots, x'_\rho \cdot x_{e_n})$ as coordinates. By Proposition 2 we know that $x'_\rho \cdot x_{e_i}$ is the probability for obtaining an outcome x_{e_i} in the selection. Clearly $x_\rho \cdot x_{e_i} = x'_\rho \cdot x_{e_i}$, which completes the proof. ■

From this it follows that the above construction is in fact a hidden measurement model for the measurement e. Along the lines of Proposition 1, one easily verifies that for the case $n = 2$, one finds Aerts' elastic model.

We still have to describe what happens for a measurement with less than n outcomes. Suppose we have l possible outcomes $e_1, \ldots, e_l$. It is possible to find a set of n mutual orthogonal points $\mathcal{X}$ such that there exist, a partition of this set in l disjoint subsets $\mathscr{X}_1, \ldots, \mathscr{X}_l$, each of them containing points representing eigenstates corresponding to the respective eigenvalues $e_1, \ldots, e_l$. Let us consider the hidden measurement model as described above with $\mathscr{X}$ as the canonical base $\{h_1, \ldots, h_n\}$. The probability to obtain an outcome e_i is the sum of the probabilities to obtain one of the $x \in \mathscr{X}_i$, which gives $\Sigma_{x \in \mathscr{X}_i} x \cdot x_\rho = x_{\mathscr{X}_i} \cdot x_\rho$ (we use the notations of Section 3). Thus one finds the same probabilities as in quantum mechanics. As the final state we find one of the eigenstates corresponding to the obtained eigenvalue. The random selection of a certain set $\mathscr{X}$ out of all the possible ones can also be considered as due to a lack of knowledge concerning the measurement.

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